Theorems of Vector Calculus Multivariable Calculus X. Du

- **Gradient Theorem** or the **Fundamental Theorem of Line Integrals**: Let *C* be a piecewise smooth curve parameterized $\vec{r}(t)$ for $t \in [t_1, t_2]$. If *f* is a differentiable function such that ∇f is continuous on *C*, then $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(t_2)) - f(\vec{r}(t_1))$.
    - Apply to conservative vector fields.
    - Line integrals of a conservative vector field are independent of path.
    - Applications:
        - Work done by gravity is independent of path
        - Conservation of Energy
- **Green's Theorem**: Let a piecewise smooth curve *C* in a plane be positively oriented, simple, and closed, and let the region *D* be the region inside *C* (i.e. $C = \partial D$).
  If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on *D*, then
  $\oint_C Pdx + Qdy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$. Note: This is just a 2D version of Stokes' Theorem
- **Stokes' Theorem** in three-space: Let *S* be an oriented surface. Let ∂S denote the oriented boundary of *S*. Let $\vec{F}$ be a vector field on *S*. Then $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$.
    - Corollary: Let S_1 and S_2 be oriented surfaces bounded by the same closed curve and induce the same orientation, and let $\vec{F}$ be a vector field on both surfaces. Then $\iint_{S_1} (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_{S_2} (\nabla \times \vec{F}) \cdot d\vec{S}$
- **Divergence Theorem**: Let *E* be a bounded, simply connected solid in three-space with a piecewise smooth surface ∂E. If $\vec{F} = \langle P, Q, R \rangle$ has continuous first-order partials, then $\oint\!\!\oint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E \nabla \cdot \vec{F} dV$
- **Stokes' Theorem** (generalized version): Let *M* be an oriented manifold and ∂M be the oriented boundary of *M*, and let ω be a differential form defined on an open set containing *M*. Then $\oint_{\partial M} \omega = \int_M d\omega$.
    - All theorems above are just specific cases of this generalized version.
    - All theorems are very intuitive to think about and understand.
- Applications
    - Mechanics
        - Work done by a conservative force (ex: gravity)
    - Electricity and magnetism
        - Gauss's Law: Apply Divergence Theorem: $\frac{Q}{\varepsilon_0} = \oint\!\!\oint_{\partial E} \vec{E} \cdot d\vec{A} = \iiint_E \nabla \cdot \vec{E} dV$
        - Ampère's Law: Apply Stokes' Theorem: $\mu_0 I = \oint_{\partial S} \vec{B} \cdot d\vec{\ell} = \iint_S (\nabla \times \vec{B}) \cdot d\vec{S}$
        - These concepts apply to all of Maxwell's equations in general.